:

Exercise 1 (OPTIONAL)

$$\mathcal{L} = \frac{C_\Sigma}{2}\dot{\phi}_A^2 + \frac{C_g}{2}(\dot{\phi}_A + \dot{\phi}_C)^2 + E_J \cos\varphi$$

1 - The equations of motion are given by Langrangian mechanics

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0 \qquad \text{where} \quad q = \phi_A \text{ in this case}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_A} = \frac{d}{dt}\left(C_\Sigma\dot{\phi}_A + C_g(\dot{\phi}_A + \dot{\phi}_C)\right)$$

$$= C_\Sigma\ddot{\phi}_A + C_g(\ddot{\phi}_A + \ddot{\phi}_C)$$

$$\frac{\partial\mathcal{L}}{\partial\phi_A} = \frac{\partial}{\partial\phi_A}\left(E_J\cos\varphi\right)$$

$$= \frac{\partial}{\partial\phi_A}\left(E_J\cos\left(\frac{\phi_A 2\pi}{\phi_0}\right)\right)$$

$$= E_J\cdot\frac{2\pi}{\phi_0}\cdot\sin\left(\frac{\phi_A 2\pi}{\phi_0}\right) = \frac{2\pi}{\phi_0}E_J\sin\varphi$$

=> EoM

$$\boxed{(C_\Sigma + C_g)\ddot{\phi}_A + C_g\ddot{\phi}_C = \frac{2\pi}{\phi_0}E_J\sin\varphi}$$

2 - The Hamiltonian of a system associated with a Langrangian is given by:

$H = p\dot{q} - \mathcal{L}$, with p the momentum $p = q_A$

and position $q = \Phi_A$

$$H = q_A \dot{\Phi}_A - \frac{C_\Sigma}{2} \dot{\Phi}_A^2 - \frac{C_g}{2} (\dot{\Phi}_A + \dot{\Phi}_c)^2 - E_J \cos \varphi$$

$$= q_A \dot{\Phi}_A - \left(\frac{C_\Sigma + C_g}{2} \right) \dot{\Phi}_A^2 - C_g \dot{\Phi}_A \dot{\Phi}_c - C_g \dot{\Phi}_c^2 - E_J \cos \varphi$$

$$= - \dot{\Phi}_A^2 \left(\frac{C_\Sigma + C_g}{2} \right) + \dot{\Phi}_A \left(q_A - C_g \dot{\Phi}_c \right) - \frac{C_g}{2} \dot{\Phi}_c^2 - E_J \cos \varphi$$

3- Neglecting the $\frac{C_g}{2} \dot{\Phi}_c^2$ term as it does not depend on the dynamics of the junction but only on the voltage bias

$$H = - \dot{\Phi}_A^2 \left(\frac{C_\Sigma + C_g}{2} \right) + \dot{\Phi}_A \left(q_A - C_g \dot{\Phi}_c \right) - E_J \cos \varphi$$

Writing

$$q_A = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_A} = C_\Sigma \dot{\Phi}_A + C_g (\dot{\Phi}_A + \dot{\Phi}_c)$$

$$\Rightarrow \quad \dot{\Phi}_A = \frac{q_A - C_g \dot{\Phi}_c}{C_\Sigma + C_g}$$

So that

$$H = - \frac{(q_A - C_g \dot{\Phi}_c)^2}{(C_\Sigma + C_g)^{\cancel{2}}} \cdot \frac{\cancel{(C_\Sigma + C_g)}}{2} + \frac{q_A - C_g \dot{\Phi}_c}{C_\Sigma + C_g} q_A - E_J \cos \varphi$$

$$H = - \frac{(q_A - C_g \dot{\Phi}_c)^2}{2 (C_\Sigma + C_g)} + \frac{(q_A - C_g \dot{\Phi}_c)^2}{C_\Sigma + C_g} - E_J \cos \varphi$$

$$H = \frac{(q_A - C_g \dot{\Phi}_c)^2}{2 (C_\Sigma + C_g)} - E_J \cos \varphi$$

Note that $\dot{\Phi}_c = V_g$ so that

$$\Rightarrow \quad \boxed{H = \frac{(q_A - C_g V_g)^2}{2 (C_\Sigma + C_g)} - E_J \cos \varphi}$$

4- q_A and Φ_A are conjugate variables so it is possible to quantize the Hamiltonian by replacing

$$q_A \to \hat{q}_A$$
$$\Phi_A \to \hat{\Phi}_A$$

Note that $\varphi = \frac{\Phi_A 2\pi}{\Phi_0} \Rightarrow \hat{\varphi} = \frac{\hat{\Phi}_A 2\pi}{\Phi_0}$

So that

$$\hat{H} = \frac{(\hat{q}_A - C_g V_g)^2}{2(C_\Sigma + C_g)} - E_J \cos\hat{\varphi}$$

5- If $C_\Sigma \gg C_g$ then we can reduce the $(C_\Sigma + C_g)$ term to just C_Σ

$$\hat{H} = \frac{(\hat{q}_A - C_g V_g)^2}{2 C_\Sigma} - E_J \cos\hat{\varphi}$$

Now replacing $\hat{q}_A = \hat{n}\, 2e$ and $q_e = n_g\, 2e$

$\Leftrightarrow C_g V_g = n_g\, 2e$

$$\hat{H} = \frac{(2e)^2 (\hat{n} - n_g)^2}{2 C_\Sigma} - E_J \cos\hat{\varphi}$$

Finally taking the charging energy $E_c = \frac{e^2}{2 C_\Sigma}$

$$\Rightarrow \boxed{\hat{H} = 4 E_c (\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}}$$

Exercise 2:

1- (OPTIONAL)

We take the power series of the exponential

$$e^{i\hat{\sigma}} = \sum_{n=0}^{\infty} \frac{(i\hat{\sigma})^n}{n!}$$

So that

$$[\hat{N}, e^{i\hat{\sigma}}] = [N, \sum_{n=0}^{\infty} \frac{(i\hat{\sigma})^n}{n!}]$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} [N, \hat{\sigma}^n]$$

Note that for $n=0$ we have $\frac{1}{1} \cdot [N, 1] = 0$

So that

$$[\hat{N}, e^{i\hat{\sigma}}] = \sum_{n=1}^{\infty} \frac{i^n}{n!} [N, \hat{\sigma}^n]$$

Now we compute $[N, \hat{\sigma}^n]$ using commutator properties of multiplication

$$[\hat{N}, \hat{\sigma}^n] = [\hat{N}, \hat{\sigma}^{n-1} \hat{\sigma}]$$

$$= \hat{\sigma}^{n-1} [\hat{N}, \hat{\sigma}] + [\hat{N}, \hat{\sigma}^{n-1}] \hat{\sigma}$$

$$= \hat{\sigma}^{n-1} \underbrace{[\hat{N}, \hat{\sigma}]}_{=-i} + \hat{\sigma}^{n-2} \underbrace{[\hat{N}, \hat{\sigma}]}_{=-i} \hat{\sigma} + [\hat{N}, \hat{\sigma}^{n-2}] \hat{\sigma}^2$$

$$= -i\hat{\sigma}^{n-1} - i\hat{\sigma}^{n-1} + [N, \hat{\sigma}^{n-2}] \hat{\sigma}^2$$

$$= \cdots$$

$$= n(-i)\,\hat{\delta}^{n-1}$$

Going back to

$$[\hat{N}, e^{i\hat{\delta}}] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \cdot n(-i)\,\hat{\delta}^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!}\,\hat{\delta}^{n-1}$$

Setting $n' = n-1$

$$= \sum_{n'=0}^{\infty} \frac{i^{n'}}{n'!}\,\hat{\delta}^{n'} = e^{i\hat{\delta}}$$

So we have that

$$[\hat{N}, e^{i\hat{\delta}}] = e^{i\hat{\delta}}$$

2- (OPTIONAL)

$$e^{i\hat{\delta}}|m\rangle = [\hat{N}, e^{i\hat{\delta}}]|m\rangle \quad \text{from question 1}$$

$$= \left(\hat{N}e^{i\hat{\delta}} - e^{i\hat{\delta}}\hat{N}\right)|m\rangle$$

$$e^{i\hat{\delta}}|m\rangle = \hat{N} e^{i\hat{\delta}}|m\rangle - e^{i\hat{\delta}}\hat{N}|m\rangle$$

$$\Leftrightarrow \quad e^{i\hat{\delta}}|m\rangle = \hat{N} e^{i\hat{\delta}}|m\rangle - m\, e^{i\hat{\delta}}|m\rangle$$

$$\Leftrightarrow \quad \hat{N}\left(e^{i\hat{\delta}}|m\rangle\right) = \left(e^{i\hat{\delta}}|m\rangle\right) + m\left(e^{i\hat{\delta}}|m\rangle\right)$$

$$\Leftrightarrow \quad \hat{N}\left(e^{i\hat{\delta}}|m\rangle\right) = (m+1)\left(e^{i\hat{\delta}}|m\rangle\right)$$

It is clear to see that $e^{i\hat{\delta}}|m\rangle$ is actually an eigenstate of the charge operator $\hat{N}$ with eigenvalue $m+1$. We know that

$$\hat{N}|m\rangle = m|m\rangle$$

So this is only possible iff $e^{i\hat{\delta}}|m\rangle = |m+1\rangle$

3-

We know that $[\hat{N}, e^{i\hat{\delta}}] = e^{i\hat{\delta}}$ and that

$$e^{i\hat{\delta}}|m\rangle = |m+1\rangle$$

To get $\hat{H}$ in the charge basis $|m\rangle$, we need to compute each matrix components:

$$\hat{H} = \sum_{m'=-\infty}^{+\infty} |m'\rangle\langle m'| \quad \hat{H} \quad \sum_{m=-\infty}^{+\infty} |m\rangle\langle m|$$

$$= \sum_{m'=-\infty}^{+\infty} |m'\rangle\langle m'| \left(4E_c(\hat{N}-n_g)^2 - E_J\cos\hat{\delta}\right) \sum_{m=-\infty}^{\infty} |m\rangle\langle m|$$

$$= \sum_{m',m=-\infty}^{\infty} \Big[|m'\rangle\langle m'| \, 4E_c \, (\hat{N} - n_g)^2 \, |m\rangle\langle m| - |m'\rangle\langle m'| \, E_J \cos\hat{\delta} \, |m\rangle\langle m| \Big]$$

First

↳ $$\langle m'| \, 4E_c \, (\hat{N} - n_g)^2 \, |m\rangle$$

$$= 4E_c \, (m - n_g)^2 \underbrace{\langle m'|m\rangle}_{=0 \text{ except for } m=m'}$$ (because n_g is not an operator)

$$= 4E_c \, (m - n_g)^2 \, \delta_{m,m'}$$

Then

↳ $$\langle m'| \, E_J \cos\hat{\delta} \, |m\rangle$$

$$= \frac{E_J}{2} \langle m'| \, e^{i\hat{\delta}} + e^{-i\hat{\delta}} \, |m\rangle$$

$$= \frac{E_J}{2} \left(\langle m'| e^{i\hat{\delta}} |m\rangle + \langle m'| e^{-i\hat{\delta}} |m\rangle \right)$$

$$= \frac{E_J}{2} \Big(\underbrace{\langle m'|m+1\rangle}_{\substack{=0 \\ \text{except if } m'=m+1}} + \underbrace{\langle m'+1|m\rangle}_{\substack{=0 \\ \text{except if } m=m'+1}} \Big)$$

$$= \frac{E_J}{2} \left(\delta_{m',m+1} + \delta_{m'+1,m} \right)$$

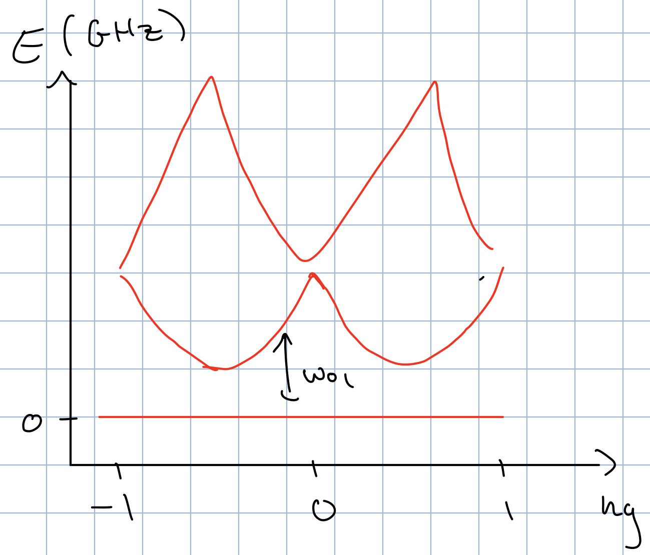
So finally

$$\hat{H} = \sum_{m',m=-\infty}^{\infty} |m'\rangle\, 4E_c\,(m+n_g)^2\,\delta_{mm'}\,\langle m| - |m'\rangle\,\frac{E_J}{2}\left(\delta_{m',m+1} + \delta_{m'+1,m}\right)\langle m|$$

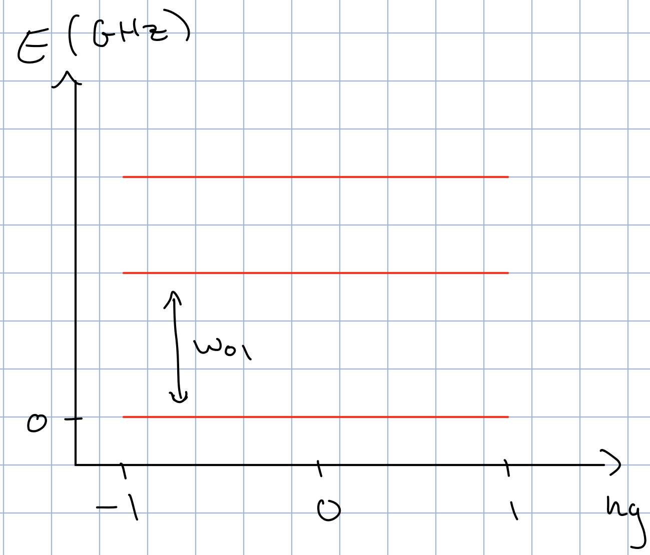
$$\boxed{\hat{H} = \sum_{m=-\infty}^{\infty} 4E_c\,(m+n_g)^2\,|m\rangle\langle m| - \frac{E_J}{2}\left(|m\rangle\langle m+1| + |m+1\rangle\langle m|\right)}$$

4 -

$\frac{E_J}{E_c} = 2$



$\frac{E_J}{E_c} = 50$



To adress our qubit, we want to send a electromagnetic pulse at the resonance frequency between $|0\rangle$ (the ground state) and $|1\rangle$ (the first excited state) ω_{01}.

If there is noise in, for example, the bias voltage, this would mean that there are random fluctuations on n_g (the x axis). If $\frac{E_J}{E_c}$ is small ($=2$) then there fluctuations has a great effect on the value of ω_{01} (resulting in an effective gate, which is not desired)

By taking a bigger ratio $\frac{E_J}{E_C}$ the lines become flat and noise in n_g does not change the resonance frequency of the qubit. That is why a Transmon is usually called "Charge noise insensitive").

Perturbation theory

1 -

$$\hat{H} = 4 E_C \hat{n}^2 - E_J \cos\hat{\varphi}$$

We expand $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

To the second order:

$$\cos\hat{\varphi} \simeq 1 - \frac{\hat{\varphi}^2}{2}$$

$$\hat{H} = 4 E_C \hat{n}^2 - E_J \left(1 - \frac{\hat{\varphi}^2}{2}\right)$$

And

$$\hat{n} = -i \left(\frac{E_J}{8E_C}\right)^{1/4} \frac{1}{\sqrt{2}} (\hat{a} - \hat{a}^+)$$

So $\hat{n}^2 = -\sqrt{\frac{E_J}{8E_C}} \frac{1}{2} (\hat{a} - \hat{a}^+)^2$

$$\hat{\varphi} = \left(\frac{2E_C}{E_J}\right)^{1/4} (\hat{a} + \hat{a}^+)$$

So $\hat{\varphi}^2 = \sqrt{\frac{2E_C}{E_J}} \left(\hat{a} + \hat{a}^+\right)^2$

$$\Rightarrow \hat{H}_0 = -4E_C \sqrt{\frac{E_J}{8E_C}} \frac{1}{2} \left(\hat{a} - \hat{a}^+\right)^2 - E_J \left(1 - \frac{1}{2}\sqrt{\frac{2E_C}{E_J}} \left(\hat{a} + \hat{a}^+\right)^2\right)$$

$$= -\sqrt{\frac{E_C E_J}{2}} \left(\cancel{\hat{a}^2} - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} + \cancel{\hat{a}^{+2}}\right) - E_J$$

$$+ \sqrt{\frac{E_C E_J}{2}} \left(\cancel{\hat{a}^2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \cancel{\hat{a}^{+2}}\right)$$

$$= \sqrt{\frac{E_C E_J}{2}} \left(2\hat{a}\hat{a}^+ + 2\hat{a}^+\hat{a}\right) - E_J$$

$$= \sqrt{2E_C E_J} \left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right) - \underbrace{E_J}_{\text{cst offset that can be neglected}}$$

By ladder operator properties:

$$\left[\hat{a}, \hat{a}^+\right] = 1$$

$$\Leftrightarrow \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

$$\Leftrightarrow \hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}$$

So that

$$\hat{H}_0 = \sqrt{2E_C E_J} \left(1 + \hat{a}^+\hat{a} + \hat{a}^+\hat{a}\right)$$

$$= \sqrt{2E_C E_J} \left(2\hat{a}^+\hat{a} + 1\right)$$

$$\Rightarrow \quad \boxed{\hat{H}_0 = \sqrt{8 E_C E_J}\left(\hat{a}^\dagger \hat{a} + 1/2\right)}$$

We recover the Hamiltonian associated with the superconducting resonator.
However, we cannot use it as a qubit because the LC resonator has equally spaced energy level so it would be impossible to specifically target the first two energy levels $|0\rangle$ and $|1\rangle$. One must add anharmonicity by way, for example, of a JJ.

2- We have that

$$E_m = -\frac{E_C}{12}\left(6m^2 + 6m + 3\right)$$

Now that we are considering higher terms in the $\cos\hat{\varphi}$

The anharmonicity is defined as how different your energy levels are between each state.
When considering a qubit (two level system) we want to confine ourselves to $|0\rangle$ and $|1\rangle$ and not adress the $|2\rangle$ state. We can define simply α by taking the difference between energy states $|0\rangle$ and $|1\rangle$ and compare with the difference between $|1\rangle$ and $|2\rangle$

$$\alpha = \frac{E_{21} - E_{10}}{\hbar}$$

$$E_{21} = E_2 - E_1$$

$$= -\frac{E_c}{12}\left(6 \cdot 2^2 + 6 \cdot 2 + 3\right) + \frac{E_c}{12}\left(6 + 6 + 3\right)$$

$$= -\frac{E_c}{12} \cdot 39 + \frac{E_c}{12} \cdot 15$$

$$= -\frac{24 E_c}{12}$$

$$= -2 E_c$$

$$E_{10} = E_1 - E_0$$

$$= -\frac{15 E_c}{12} + \frac{E_c}{12} \cdot 3$$

$$= -\frac{12 E_c}{12}$$

$$= -E_c$$

So that

$$\hbar \alpha = -2 E_c + E_c$$

$$\Rightarrow \quad \boxed{\alpha = -\frac{E_c}{\hbar}}$$

So to get the best "two level system" we want the highest α possible which is directly proportional to E_C, so we need to aim for large E_C. As the charging energy is inversly proportional to the qubit capacitance, we want to design small values of C.

But we also want to be in the transmon limit so a ratio $\frac{E_J}{E_C} \gg 1 \Leftrightarrow E_J \gg E_C$. The E_J is usually fixed by the junction area which can't be infinitely small or large so it is better to play with the capacitance and reduce E_C (bigger C).

As you can see, there is a competition between the two regimes and a right balance needs to be achieved. The only reason why transmons are possible to be built is because by reducing E_C, we are reaching charge insensitivity exponentially while only loosing anharmonicity polonomially.

Exercise 3:

Orange: the transmon qubit (SQUID are too small to see) used for storing the quantum information

Light Blue: Coupling resonators for communication between qubits (entanglement, 2qubit gates)

Green: Flux line to apply magnetic field to the SQUID and change the qubits resonant frequency

Grey: Drive line to send microwave pulses and apply single qubit gates

Red: Readout resonator to readout the qubit state, we send a pulse in the RO resonator dispersively coupled to the qubit. The pulse will carry the qubit information back out.

Purple: Purcell filters a specially lossy resonator (wide bandgap) to easily let the microwave pulse for RO pass though but strongly impedes propagation of photons at the qubit frequency. It is a filter for protecting the qubit information

Dark Purple: Feedline : Resonator to which the measurement instruments are directly connected to. The feedline is connected to multiple qubits for multiple ROs using only 2 outside ports. This is called multiplexing and only works if the qubits are at different frequencies.